# Lagrangian BV quantization and Ward identities

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The Ward identities are the relations which the complete Green functions of quantum fields satisfy if an original classical Lagrangian system is degenerate. A generic degenerate Lagrangian system of even and odd fields is considered. It is characterized by a hierarchy of reducible Noether identities and gauge supersymmetries parameterized by antifields and ghosts, respectively. In the framework of the BV quantization procedure, an original degenerate Lagrangian is extended to ghosts and antifields in order to satisfy the master equation. Replacing antifields with gauge fixing terms, one comes to a non-degenerate Lagrangian which is quantized in the framework of perturbed QFT. This Lagrangian possesses a BRST symmetry. The corresponding Ward identities are obtained. They generalize Ward identities in the Yang–Mills gauge theory to a general case of reducible gauge supersymmetries depending on derivatives of fields of any order. A supersymmetric Yang-Mills model is considered.

## I. INTRODUCTION

In a general setting, by Ward identities are meant the relations which the complete Green functions of quantum fields satisfy if an original classical Lagrangian system is degenerate.

A generic degenerate Lagrangian system of even and odd fields is considered [1, 2, 3]. Its Euler–Lagrange operator satisfies nontrivial Noether identities. They need not be independent, but obey the first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. Being finitely generated, the Noether and higher-stage Noether identities are parameterized by the modules of antifields (Section III). The Noether second theorem states the relation between these Noether identities and the reducible gauge supersymmetries of a degenerate Lagrangian system parameterized by ghosts (Section IV).

In the framework of the BV quantization of a degenerate Lagrangian system [4, 5, 6], its original Lagrangian $L$ is extended to the above mentioned ghosts and antifields in order to satisfy the so-called classical master equation (Section V). Replacing antifields with gauge fixed terms, one comes to a non-degenerate gauge-fixing Lagrangian $L_{GF}$ (61) which can

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be quantized in the framework of perturbed QFT (Section VII). This Lagrangian possesses the gauge-fixing BRST symmetry $\widehat{u}$ (62). We obtain the corresponding Ward identities (82) – (83) for complete Green functions of quantum fields (Section VIII). They generalize the Ward identities in the Yang–Mills gauge theory [7, 8], whose gauge symmetries are irreducible, independent of field derivatives, and they form a finite-dimensional Lie algebra.

Note that the BRST transformation $\widehat{u}$ is necessary non-linear and, moreover, it need not maintain the measure term in a generating functional of Green functions. Therefore, the Ward identities (82) – (83) contain an anomaly in general. Another anomaly can issue from the fact that, after regularization, a Lagrangian $L_{GF}$ need not be BRST invariant. Here, we however do not concern a regularization procedure.

In Section IX, an example of supersymmetric Yang-Mills model is considered.

## II. GRASSMANN-GRADED LAGRANGIAN SYSTEMS

Bearing in mind the BV quantization, we consider a Lagrangian field system on $X = \mathbb{R}^n$, $2 \leq n$, coordinated by $(x^\lambda)$. Such a Lagrangian system is algebraically described in terms of a certain bigraded differential algebra (henceforth BGDA) $\mathcal{P}^*[Q;Y]$ [1, 2, 3]. Unless otherwise stated, by a gradation throughout is meant $\mathbb{Z}_2$-gradation.

Let $Y \to X$ be an affine bundle coordinated by $(x^\lambda, y^i)$ whose sections are even classical fields, and let $Q \to X$ be a vector bundle coordinated by $(x^\lambda, q^a)$ whose sections are odd classical fields. Let $J^rY \to X$ and $J^rQ \to X$, $r = 1, \ldots,$ be the corresponding $r$-order jet bundles, endowed with the adapted coordinates $(x^\lambda, y^i_\Lambda)$ and $(x^\lambda, q^a_\Lambda)$, respectively, where $\Lambda = (\lambda_1 ... \lambda_k)$, $|\Lambda| = k$, $k = 1, \ldots, r$, are symmetric multi-indices. The index $r = 0$ conventionally stands for $Y$ and $Q$. For each $r = 0, \ldots,$ we consider a graded manifold $(X, \mathcal{A}_{J^rQ})$, whose body is $X$ and the ring of graded functions consists of sections of the exterior bundle

$$\wedge(J^rQ)^* = \mathbb{R} \underset{X}{\oplus} (J^rQ)^* \underset{X}{\oplus} \overset{2}{\wedge}(J^rQ)^* \underset{X}{\oplus} \cdots,$$

where $(J^rQ)^*$ is the dual of a vector bundle $J^rQ \to X$. The global basis for $(X, \mathcal{A}_{J^rQ})$ is $\{x^\lambda, c^a_\Lambda\}$, $|\Lambda| = 0, \ldots, r$. Let us consider the graded commutative $C^\infty(X)$-ring $\mathcal{P}^0[Q;Y]$ generated by the even elements $y^i_\Lambda$ and the odd ones $c^a_\Lambda$, $|\Lambda| \geq 0$. The collective symbols $s^A_\Lambda$ further stand for these elements, together with the symbol $[A] = [s^A_\Lambda]$ for their Grassmann parity. In fact, $\mathcal{P}^0[Q;Y]$ is the $C^\infty(X)$-ring of polynomials in the graded elements $s^a_\Lambda$.

Let $\mathfrak{d}\mathcal{P}^0[Q;Y]$ be the Lie superalgebra of (left) graded derivations of the $\mathbb{R}$-ring $\mathcal{P}^0[Q;Y]$, i.e.,

$$u(ff') = u(f)f' + (-1)^{[u][f]} f u(f'), \qquad f, f' \in \mathcal{P}^0[Q;Y], \qquad u \in \mathfrak{d}\mathcal{P}^0[Q;Y].$$

Its elements take the form

$$u = u^\lambda \partial_\lambda + \sum_{0\leq|\Lambda|} u^A_\Lambda \partial^\Lambda_A, \qquad u^\lambda, u^A_\Lambda \in \mathcal{P}^0[Q;Y], \tag{1}$$

where $\partial^\Lambda_A(s^B_\Sigma) = \delta^B_A \delta^\Lambda_\Sigma$ up to permutations of multi-indices $\Lambda$ and $\Sigma$. By a summation over a multi-index $\Lambda$ is meant separate summation over each index $\lambda_i$. For instance, we have the total derivatives

$$d_\lambda = \partial_\lambda + \sum_{0\leq|\Lambda|} s^A_{\lambda+\Lambda} \partial^\Lambda_A \in \mathfrak{d}\mathcal{P}^0[Q;Y],$$

where $\lambda + \Lambda$ denotes the multi-index $(\lambda, \lambda_1, \ldots, \lambda_k)$.

With the Lie superalgebra $\mathfrak{d}\mathcal{P}^0[Q;Y]$, one can construct the minimal Chevalley–Eilenberg differential calculus

$$0 \to \mathbb{R} \to \mathcal{P}^0[Q;Y] \xrightarrow{d} \mathcal{P}^1[Q;Y] \xrightarrow{d} \cdots \mathcal{P}^2[Q;Y] \xrightarrow{d} \cdots$$

over the ring $\mathcal{P}^0[Q;Y]$. It is a desired BGDA $\mathcal{P}^*[Q;Y]$. Its elements $\phi \in \mathcal{P}^k[Q;Y]$ are graded $\mathcal{P}^0[Q;Y]$-linear $k$-forms on $\mathfrak{d}\mathcal{P}^0[Q;Y]$ with values in $\mathcal{P}^0[Q;Y]$. The graded exterior product $\wedge$ and the even Chevalley–Eilenberg coboundary operator $d$, called the graded exterior differential, obey the relations

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'|+[\phi][\phi']} \phi' \wedge \phi, \qquad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|} \phi \wedge d\phi',$$

where $|.|$ denotes the form degree. Since $\mathcal{P}^*[Q;Y]$ is a minimal differential calculus over $\mathcal{P}^0[Q;Y]$, it is generated by the elements $dx^\lambda$, $ds^A_\Lambda$ dual of $\partial_\lambda$, $\partial^\Lambda_A$, i.e.,

$$\phi = \sum \phi^{\Lambda_1\ldots\Lambda_r}_{A_1\ldots A_r \lambda_1\ldots\lambda_k} ds^{A_1}_{\Lambda_1} \wedge \cdots \wedge ds^{A_r}_{\Lambda_r} \wedge dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k}, \qquad \phi \in \mathcal{P}^0[Q;Y].$$

In particular, the graded exterior differential takes the familiar form

$$d\phi = dx^\lambda \wedge \partial_\lambda \phi + \sum_{0\leq|\Lambda|} ds^A_\Lambda \wedge \partial^\Lambda_A \phi.$$

Let $\mathcal{O}^*X$ be the graded differential algebra of exterior forms on $X$. There is the natural monomorphism $\mathcal{O}^*X \to \mathcal{P}^*[Q;Y]$.

Given a graded derivation $u$ (1) of the $\mathbb{R}$-ring $\mathcal{P}^0[Q;Y]$, the interior product $u\rfloor\phi$ and the Lie derivative $\mathbf{L}_u\phi$, $\phi \in \mathcal{P}^*[Q;Y]$, are defined by the formulas

$$u\rfloor\phi = u^\lambda \phi_\lambda + \sum_{0\leq|\Lambda|} (-1)^{[\phi^\Lambda_A][A]} u^A_\Lambda \phi^\Lambda_A, \qquad \phi \in \mathcal{P}^1[Q;Y],$$

$$u\rfloor(\phi \wedge \sigma) = (u\rfloor\phi) \wedge \sigma + (-1)^{|\phi|+[\phi][u]} \phi \wedge (u\rfloor\sigma), \qquad \phi, \sigma \in \mathcal{P}^*[Q;Y],$$

$$\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \qquad \mathbf{L}_u(\phi \wedge \sigma) = \mathbf{L}_u(\phi) \wedge \sigma + (-1)^{[u][\phi]} \phi \wedge \mathbf{L}_u(\sigma).$$

For instance, let us denote $d_\lambda\phi = \mathbf{L}_{d_\lambda}\phi$ and $d_\Lambda = d_{\lambda_1}\cdots d_{\lambda_k}$. Given graded functions $f^\Lambda$, $f'$ and a graded form $\Phi$, there are useful relations

$$\sum_{0\leq|\Lambda|\leq k} f^\Lambda d_\Lambda f' d^n x = \sum_{0\leq|\Lambda|\leq k} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda) f' d^n x + d_H\sigma, \tag{2}$$

$$\sum_{0\leq|\Lambda|\leq k} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda\Phi) = \sum_{0\leq|\Lambda|\leq k} \eta(f)^\Lambda d_\Lambda\Phi, \tag{3}$$

$$\eta(f)^\Lambda = \sum_{0\leq|\Sigma|\leq k-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C^{|\Sigma|}_{|\Sigma+\Lambda|} d_\Sigma f^{\Sigma+\Lambda}, \qquad C^a_b = \frac{b!}{a!(b-a)!}, \tag{4}$$

$$(\eta\circ\eta)(f)^\Lambda = f^\Lambda.$$

The BGDA $\mathcal{P}^*[Q;Y]$ is decomposed into $\mathcal{P}^0[Q;Y]$-modules $\mathcal{P}^{k,r}[Q;Y]$ of $k$-contact and $r$-horizontal graded forms

$$\phi = \sum_{0\leq|\Lambda_i|} \phi^{\Lambda_1\ldots\Lambda_k}_{A_1\ldots A_k\mu_1\ldots\mu_r} \theta^{A_1}_{\Lambda_1}\wedge\cdots\wedge\theta^{A_k}_{\Lambda_k}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}, \qquad \theta^A_\Lambda = ds^A_\Lambda - s^A_{\lambda+\Lambda}dx^\lambda,$$

$$h_k : \mathcal{P}^*[Q;Y]\to\mathcal{P}^{k,*}[Q;Y], \qquad h^r : \mathcal{P}^*[Q;Y]\to\mathcal{P}^{*,r}[Q;Y].$$

Accordingly, the graded exterior differential on $\mathcal{P}^*[Q;Y]$ falls into the sum $d = d_V + d_H$ of the vertical and total differentials where $d_H\phi = dx^\lambda\wedge d_\lambda\phi$.

A graded derivation $u$ (1) is called contact if the Lie derivative $\mathbf{L}_u$ preserves the ideal of contact graded forms of the BGDA $\mathcal{P}^*[Q;Y]$. Further, we restrict our consideration to vertical contact graded derivations, vanishing on $\mathcal{O}^*X$. Such a derivation

$$\vartheta = \upsilon^A\partial_A + \sum_{0<|\Lambda|} d_\Lambda\upsilon^A\partial^\Lambda_A \tag{5}$$

is determined by its first summand $\upsilon = \upsilon^A\partial_A$, called a generalized vector field. The relations

$$\vartheta\rfloor d_H\phi = -d_H(\vartheta\rfloor\phi), \qquad \mathbf{L}_\vartheta(d_H\phi) = d_H(\mathbf{L}_\vartheta\phi), \qquad \phi\in\mathcal{P}^*[Q;Y],$$

hold. A vertical contact graded derivation $\vartheta$ (5) is called nilpotent if $\mathbf{L}_\vartheta(\mathbf{L}_\vartheta\phi) = 0$ for any horizontal graded form $\phi\in\mathcal{P}^{0,*}[Q;Y]$. One can show that $\vartheta$ (5) is nilpotent only if it is odd and iff all $\upsilon^A$ obey the equality

$$\vartheta(\upsilon) = \vartheta(\upsilon^A\partial_A) = \sum_{0\leq|\Sigma|} \upsilon^B_\Sigma\partial^\Sigma_B(\upsilon^A)\partial_A = 0. \tag{6}$$

Lagrangian systems are described in terms of the BGDA $\mathcal{P}^*[Q;Y]$ and its graded derivations as follows [1, 2, 3]. The differentials $d_H$ and $d_V$, the projector

$$\varrho = \sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n, \qquad \overline{\varrho}(\phi) = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda(\partial^\Lambda_A\rfloor\phi)], \qquad \phi\in\mathcal{P}^{>0,n}[Q;Y],$$

and the graded variational operator $\delta = \varrho \circ d$ split the BGDA $\mathcal{P}^*[Q;Y]$ into the graded variational bicomplex [1, 2, 9]. We restrict our consideration to its short variational subcomplex

$$0 \to \mathbb{R} \longrightarrow \mathcal{P}^0[Q;Y] \xrightarrow{d_H} \mathcal{P}^{0,1}[Q;Y] \cdots \xrightarrow{d_H} \mathcal{P}^{0,n}[Q;Y] \xrightarrow{\delta} \mathbf{E}_1 = \varrho(\mathcal{P}^{1,n}[Q;Y]). \tag{7}$$

One can think of its even elements

$$L = \mathcal{L}d^n x \in \mathcal{P}^{0,n}[Q;Y], \tag{8}$$

$$\delta L = \theta^A \wedge \mathcal{E}_A d^n x = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A L)d^n x \in \mathbf{E}_1 \tag{9}$$

as being a graded Lagrangian and its Euler–Lagrange operator, respectively. They possess the following properties.

(i) The complex (7) is exact at all the terms, except $\mathbb{R}$. In particular, any $\delta$-closed (i.e., variationally trivial) graded density $L \in \mathcal{P}^{0,n}[Q;Y]$ is $d_H$-exact.

(ii) The identity $(\delta\circ\delta)(L) = 0$ leads to the useful equalities

$$\eta(\partial_B\mathcal{E}_A)^\Lambda = (-1)^{[A][B]}\partial^\Lambda_A\mathcal{E}_B. \tag{10}$$

(iii) By virtue of the formula (2), the Lie derivative $\mathbf{L}_\vartheta L$ of a Lagrangian $L$ along a vertical contact graded derivation $\vartheta$ (5) admits the decomposition

$$\mathbf{L}_\vartheta L = \upsilon\rfloor\delta L + d_H\sigma. \tag{11}$$

One says that an odd vertical contact graded derivation $\vartheta$ (5) is a variational supersymmetry of a Lagrangian $L$ if the Lie derivative $\mathbf{L}_\vartheta L$ is $d_H$-exact or, equivalently, the odd graded density $\upsilon\rfloor\delta L = \upsilon^A\mathcal{E}_A d^n x$ is $d_H$-exact.

For the sake of simplicity, the common symbol $\upsilon$ further stands for a generalized vector field $\upsilon$, the vertical contact graded derivation $\vartheta$ (5) determined by $\upsilon$ and the Lie derivative $\mathbf{L}_\vartheta$. We agree to call all these operators a graded derivation of the BGDA $\mathcal{P}^*[Q;Y]$.

One also deals with right contact graded derivations $\overleftarrow{\upsilon} = \overleftarrow{\partial}_A\upsilon^A$ of the BGDA $\mathcal{P}^*[Q;Y]$. They act on graded forms $\phi$ on the right by the rule

$$\overleftarrow{\upsilon}(\phi) = \overleftarrow{\mathrm{d}}(\phi)\lfloor\overleftarrow{\upsilon} + \overleftarrow{\mathrm{d}}(\phi\lfloor\overleftarrow{\upsilon}), \qquad \overleftarrow{\upsilon}(\phi\wedge\phi') = (-1)^{[\phi'][\overleftarrow{\upsilon}]}\,\overleftarrow{\upsilon}(\phi)\wedge\phi' + \phi\wedge\overleftarrow{\upsilon}(\phi').$$

For instance,

$$\overleftarrow{\partial}_A(\phi) = (-1)^{([\phi]+1)[A]}\partial_A(\phi), \qquad \overleftarrow{\mathrm{d}}_\Lambda = d_\Lambda, \qquad \overleftarrow{\mathrm{d}}_H(\phi) = (-1)^{|\phi|}d_H(\phi).$$

With right graded derivations, one obtains the right Euler–Lagrange operator

$$\overleftarrow{\delta}\, L = \overleftarrow{\mathcal{E}}_A d^n x \wedge \theta^A, \qquad \overleftarrow{\mathcal{E}}_A = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}d_\Lambda(\overleftarrow{\partial}\,{}^\Lambda_A(L)).$$

An odd right graded derivation $\overleftarrow{\upsilon}$ is a variational supersymmetry of a Lagrangian $L$ iff the odd graded density $\overleftarrow{\mathcal{E}}_A \upsilon^A d^n x$ is $d_H$-exact.

## III. NOETHER IDENTITIES AND ANTIFIELDS

Given a graded Lagrangian $L$ (8), one can associate to its Euler–Lagrange operator $\delta L$ (9) the exact chain Koszul–Tate complex with the boundary operator whose nilpotency conditions provide the Noether and higher-stage Noether identities for $\delta L$ [3].

Let us start with the following notation. Given vector bundles $V, V', E, E'$ over $X$, we consider the BGDA

$$\mathcal{P}^*[V'V;Q;Y;EE'] = \mathcal{P}^*[V' \underset{X}{\times} V \underset{X}{\times} Q; Y \underset{X}{\times} E \underset{X}{\times} E'].$$

By a density-dual of a vector bundle $E \to X$ is meant

$$\overline{E}^* = E^* \otimes_X \overset{n}{\wedge} T^*X.$$

By $\overline{Y}^*$ is denoted the density-dual of the vector bundle which an affine bundle $Y$ is modelled on.

Let us extend the BGDA $\mathcal{P}^*[Q;Y]$ to the BGDA $\mathcal{P}^*[\overline{Y}^*;Q;Y;\overline{Q}^*]$ whose basis is $\{s^A, \overline{s}_A\}$, where $[\overline{s}_A] = ([A]+1)\mathrm{mod}\,2$. We call $\overline{s}_A$ the antifields of antifield number $\mathrm{Ant}[\overline{s}_A] = 1$. The BGDA $\mathcal{P}^*[\overline{Y}^*;Q;Y;\overline{Q}^*]$ is provided with the nilpotent right graded derivation $\overline{\delta} = \overleftarrow{\partial}{}^A \mathcal{E}_A$, where $\mathcal{E}_A$ are the graded variational derivatives (9). We call $\overline{\delta}$ the Koszul–Tate differential, and say that an element $\phi \in \mathcal{P}^*[\overline{Y}^*;Q;Y;\overline{Q}^*]$ vanishes on the shell if it is $\overline{\delta}$-exact, i.e., $\phi = \overline{\delta}\sigma$. Let us consider the module $\mathcal{P}^{0,n}[\overline{Y}^*;Q;Y;\overline{Q}^*]$ of graded densities. It contains the chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}[\overline{Y}^*;Q;Y;\overline{Q}^*]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}[\overline{Y}^*;Q;Y;\overline{Q}^*]_2. \tag{12}$$

This complex is exact at $\mathrm{Im}\,\overline{\delta}$. Let us consider its first homology $H_1(\overline{\delta})$.

A generic one-chain of the complex (12) takes the form

$$\Phi = \sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} \overline{s}_{\Lambda A} d^n x, \qquad \Phi^{A,\Lambda} \in \mathcal{P}^0[Q;Y]. \tag{13}$$

Then the cycle condition $\overline{\delta}\Phi = 0$ provides a reduction condition

$$\sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A d^n x = 0 \tag{14}$$

on the graded variational derivatives $\mathcal{E}_A$. Conversely, any reduction condition of form (14) comes from some cycle (13). The reduction condition (14) is trivial if a cycle is a boundary, i.e., it takes the form

$$\Phi = \sum_{0\leq|\Lambda|,|\Sigma|} T^{(A\Lambda)(B\Sigma)} d_\Sigma \mathcal{E}_B \overline{s}_{\Lambda A} d^n x, \qquad T^{(A\Lambda)(B\Sigma)} = -(-1)^{[A][B]} T^{(B\Sigma)(A\Lambda)}.$$

A Lagrangian system is called degenerate if there exist nontrivial reduction conditions (14), called the Noether identities.

One can say something more if the $\mathcal{P}^0[Q;Y]$-module $H_1(\overline{\delta})$ is finitely generated, i.e., there are elements of $H_1(\overline{\delta})$ making up a free graded $C^\infty(X)$-module $\mathcal{C}_{(0)}$ of finite rank. By virtue of the Serre–Swan theorem [3], $\mathcal{C}_{(0)}$ is isomorphic to the module of sections of the product $\overline{V}^* \underset{X}{\times} \overline{E}^*$ where $\overline{V}^*$, $\overline{E}^*$ are the density-duals of some vector bundles $V \to X$ and $E \to X$. Let $\{\Delta_r\}$ be a basis for this $C^\infty(X)$-module. Every element $\Phi \in H_1(\overline{\delta})$ factorizes

$$\Phi = \sum_{0\leq|\Xi|} G^{r,\Xi} d_\Xi \Delta_r d^n x, \qquad \Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} \overline{s}_{\Lambda A}, \tag{15}$$

via elements $\Delta_r \in \mathcal{C}_{(0)}$, called the Noether operators. Accordingly, any Noether identity (14) is a corollary of the identities

$$\sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{16}$$

called the complete Noether identities. In this case, the complex (12) can be extended to a one-exact complex with a boundary operator whose nilpotency conditions are equivalent to the Noether identities (16).

Note that, if there is no danger of confusion, elements of homology are identified to its representatives. A chain complex is called $r$-exact if its homology of degree $k \leq r$ is trivial.

Let us enlarge the BGDA $\mathcal{P}^*[\overline{Y}^*;Q;Y;\overline{Q}^*]$ to the BGDA $\mathcal{P}^*[\overline{E}^*\overline{Y}^*;Q;Y;\overline{Q}^*\overline{V}^*]$ possessing the basis $\{s^A, \overline{s}_A, \overline{c}_r\}$ where $[\overline{c}_r] = ([\Delta_r]+1) \bmod 2$ and $\mathrm{Ant}[\overline{c}_r] = 2$. This BGDA is provided with the nilpotent right graded derivation $\delta_0 = \overline{\delta} + \overleftarrow{\partial}^r \Delta_r$. Its nilpotency conditions (6) are equivalent to the Noether identities (16). Then the module $\mathcal{P}^{0,n}[\overline{E}^*\overline{Y}^*;Q;Y;\overline{Q}^*\overline{V}^*]_{\leq 3}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq 3$ is split into the chain complex

$$\begin{array}{l} 0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}[\overline{Y}^*;Q;Y;\overline{Q}^*]_1 \xleftarrow{\delta_0} \mathcal{P}^{0,n}[\overline{E}^*\overline{Y}^*;Q;Y;\overline{Q}^*\overline{V}^*]_2 \\ \qquad \xleftarrow{\delta_0} \mathcal{P}^{0,n}[\overline{E}^*\overline{Y}^*;Q;Y;\overline{Q}^*\overline{V}^*]_3. \end{array} \tag{17}$$

Let $H_*(\delta_0)$ be its homology. We have $H_0(\delta_0) = H_0(\overline{\delta}) = 0$. Furthermore, any one-cycle $\Phi$ up to a boundary takes the form (15) and, therefore, it is a $\delta_0$-boundary

$$\Phi = \sum_{0\leq|\Sigma|} G^{r,\Xi} d_\Xi \Delta_r d^n x = \delta_0 (\sum_{0\leq|\Sigma|} G^{r,\Xi} \overline{c}_{\Xi r} d^n x).$$

Hence, the complex (17) is one-exact.

Turn now to the homology $H_2(\delta_0)$ of the complex (17). A generic two-chain reads

$$\Phi = G + H = \sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r} d^n x + \sum_{0\leq|\Lambda|,|\Sigma|} H^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{s}_{\Sigma B} d^n x. \tag{18}$$

The cycle condition $\delta_0\Phi = 0$ provides the reduction condition

$$\sum_{0\leq|\Lambda|} G^{r,\Lambda} d_\Lambda \Delta_r d^n x + \overline{\delta} H = 0 \tag{19}$$

on the Noether operators (15). Conversely, let

$$\Phi = \sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r} d^n x \in \mathcal{P}^{0,n}[\overline{E}^*\overline{Y}^*; Q; Y; \overline{Q}^*\overline{V}^*]_2$$

be a graded density such that the reduction condition (19) holds. Obviously, this reduction condition is a cycle condition of the two-chain (18). It is trivial either if a two-cycle $\Phi$ (18) is a boundary or its summand $G$, linear in antifields, vanishes on the shell.

A degenerate Lagrangian system is said to be one-stage reducible if there exist nontrivial reduction conditions (19), called the first-stage Noether identities. One can show that first-stage Noether identities are identified to elements of the homology $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi \in \mathcal{P}^{0,n}[\overline{Y}^*; Q; Y; \overline{Q}^*]_2$ is a $\delta_0$-boundary [3]. Furthermore, if first-stage Noether identities are finitely generated, the complex (17) is extended to a two-exact complex with a boundary operator whose nilpotency conditions are equivalent to complete Noether and first-stage Noether identities. If the third homology of this complex is not trivial, we have second-stage Noether identities, and so on. Iterating the arguments, one says that a degenerate Lagrangian system $(\mathcal{P}^*[Q;Y], L)$ is $N$-stage reducible if the following holds [3].

(i) There exist vector bundles $V_1, \ldots, V_N, E_1, \ldots, E_N$ over $X$, and the BGDA $\mathcal{P}^*[Q;Y]$ is enlarged to the BGDA

$$\overline{\mathcal{P}}^*\{N\} = \mathcal{P}^*[\overline{E}^*_N \cdots \overline{E}^*_1 \overline{E}^* \overline{Y}^*; Q; Y; \overline{Q}^* \overline{V}^* \overline{V}^*_1 \cdots \overline{V}^*_N] \tag{20}$$

with the basis $\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}\}$ graded by antifield numbers $\mathrm{Ant}[\overline{c}_{r_k}] = k+2$. Let the indexes $k = -1, 0$ further stand for $\overline{s}_A$ and $\overline{c}_r$, respectively.

(ii) The BGDA $\overline{\mathcal{P}}^*\{N\}$ (20) is provided with a nilpotent graded derivation

$$\delta_N = \overleftarrow{\partial}^A \mathcal{E}_A + \sum_{0\leq|\Lambda|} \overleftarrow{\partial}^r \Delta^{A,\Lambda}_r \overline{s}_{\Lambda A} + \sum_{1\leq k\leq N} \overleftarrow{\partial}^{r_k} \Delta_{r_k}, \tag{21}$$

$$\Delta_{r_k} = G_{r_k} + h_{r_k} = \sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} \overline{c}_{\Lambda r_{k-1}} + \sum_{0\leq|\Sigma|,|\Xi|} (h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} + \ldots), \tag{22}$$

of antifield number -1. It is called the $N$-stage Koszul–Tate differential.

(iii) With $\delta_N$, the module $\overline{\mathcal{P}}^{0,n}\{N\}_{\leq N+3}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq N+3$ is split into the $(N+2)$-exact chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}[\overline{Y}^*;Q;Y;\overline{Q}^*]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}\{1\}_3 \cdots \tag{23}$$
$$\xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}^{0,n}\{N-1\}_{N+1} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}\{N\}_{N+2} \xleftarrow{\delta_N} \overline{\mathcal{P}}^{0,n}\{N\}_{N+3},$$

which satisfies the homology regularity condition. This condition states that, any $\delta_{k<N-1}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}\{k\}_{k+3} \subset \overline{\mathcal{P}}^{0,n}\{k+1\}_{k+3}$ is a $\delta_{k+1}$-boundary.

(iv) The nilpotency of $\delta_N$ implies the Noether identities (16) and the $k$-stage Noether identities

$$\sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} d_\Lambda(\sum_{0\leq|\Sigma|} \Delta^{r_{k-2},\Sigma}_{r_{k-1}}\overline{c}_{\Sigma r_{k-2}}) + \overline{\delta}(\sum_{0\leq|\Sigma|,|\Xi|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}) = 0 \tag{24}$$

for $k=1,\ldots,N$. Accordingly, $\Delta_{r_k}$ (22) are called the $k$-stage Noether operators.

## IV. GAUGE SUPERSYMMETRIES AND GHOSTS

The Noether second theorem associates to the Koszul–Tate complex (23) the sequence (36) graded in ghosts. Its ascent operator $\upsilon_e$ (35) provides the gauge and higher-stage gauge supersymmetries of an original Lagrangian $L$.

Given the BGDA $\overline{\mathcal{P}}^*\{N\}$ (20), let us consider the BGDA

$$\mathcal{P}^*\{N\} = \mathcal{P}^*[V_N\cdots V_1V;Q;Y;EE_1\cdots E_N] \tag{25}$$

with the basis $\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}\}$ and the BGDA

$$P^*\{N\} = \mathcal{P}^*[\overline{E}^*_N\cdots\overline{E}^*_1\overline{E}^*V_N\cdots V_1V\overline{Y}^*;Q;Y;\overline{Q}^*EE_1\cdots E_N\overline{V}^*\overline{V}^*_1\cdots\overline{V}^*_N] \tag{26}$$

with the basis

$$\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}\}, \tag{27}$$

where $[c^{r_k}] = ([\overline{c}_{r_k}]+1)\mathrm{mod}\,2$ and $\mathrm{Ant}[c^{r_k}] = -(k+1)$. We call $c^{r_k}$, $k=0,\ldots,N$, the ghosts of ghost number $\mathrm{gh}[c^{r_k}] = k+1$. Clearly, the BGDAs $\overline{\mathcal{P}}^*\{N\}$ (20) and $\mathcal{P}^*\{N\}$ (25) are subalgebras of the BGDA $P^*\{N\}$ (26). The $N$-stage Koszul–Tate differential $\delta_N$ (21) is naturally prolonged to a graded derivation of the BGDA $P^*\{N\}$ (26).

Let us extend an original Lagrangian $L$ to the even graded density

$$L_e = \mathcal{L}_e d^nx = L + L_1 = L + \sum_{0\leq k\leq N} c^{r_k}\Delta_{r_k}d^nx = L + \delta_N(\sum_{0\leq k\leq N} c^{r_k}\overline{c}_{r_k}d^nx), \tag{28}$$

of zero antifield number where $L_1$ is linear in ghosts. It is readily observed that $\delta_N(L_e)=0$, i.e., $\delta_N$ is a variational supersymmetry of the Lagrangian $L_e$ (28). It follows that

$$[\frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{s}_A}\mathcal{E}_A+\sum_{0\leq k\leq N}\frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{c}_{r_k}}\Delta_{r_k}]d^nx=[\upsilon^A\mathcal{E}_A+\sum_{0\leq k\leq N}\upsilon^{r_k}\frac{\delta\mathcal{L}_e}{\delta c^{r_k}}]d^nx=d_H\sigma, \tag{29}$$

$$\upsilon^A=\frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{s}_A}=u^A+w^A=\sum_{0\leq|\Lambda|}c^r_\Lambda\eta(\Delta^A_r)^\Lambda+\sum_{1\leq i\leq N}\sum_{0\leq|\Lambda|}c^{r_i}_\Lambda\eta(\overleftarrow{\partial}^A(h_{r_i}))^\Lambda,$$

$$\upsilon^{r_k}=\frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{c}_{r_k}}=u^{r_k}+w^{r_k}=\sum_{0\leq|\Lambda|}c^{r_{k+1}}_\Lambda\eta(\Delta^{r_k}_{r_{k+1}})^\Lambda+\sum_{k+1<i\leq N}\sum_{0\leq|\Lambda|}c^{r_i}_\Lambda\eta(\overleftarrow{\partial}^{r_k}(h_{r_i}))^\Lambda,$$

(see the formulas (3) – (4)). The equality (29) falls into the set of equalities

$$\frac{\overleftarrow{\delta}(c^r\Delta_r)}{\delta\overline{s}_A}\mathcal{E}_Ad^nx=u^A\mathcal{E}_Ad^nx=d_H\sigma_0, \tag{30}$$

$$[\frac{\overleftarrow{\delta}(c^{r_i}\Delta_{r_i})}{\delta\overline{s}_A}\mathcal{E}_A+\sum_{k<i}\frac{\overleftarrow{\delta}(c^{r_i}\Delta_{r_i})}{\delta\overline{c}_{r_k}}\Delta_{r_k}]d^nx=d_H\sigma_i, \qquad i=1,\ldots,N. \tag{31}$$

It follows from the equality (30) and formula (11) that the graded derivation

$$u=u^A\frac{\partial}{\partial s^A}, \qquad u^A=\sum_{0\leq|\Lambda|}c^r_\Lambda\eta(\Delta^A_r)^\Lambda, \tag{32}$$

is a variational supersymmetry of an original Lagrangian $L$. This variational supersymmetry is parameterized by ghosts $c^r$, i.e., it is a gauge supersymmetry of $L$ [1, 2]. Every equality (31) is split into a set of equalities with respect to the polynomial degree in antifields. Let us consider the one, linear in antifields $\overline{c}_{r_{i-2}}$ and their jets (where by $\overline{c}_{r_{-1}}$ are meant $\overline{s}_A$). It is brought into the form

$$[\sum_{0\leq|\Xi|}(-1)^{|\Xi|}d_\Xi(c^{r_i}\sum_{0\leq|\Sigma|}h^{(r_{i-2},\Sigma)(A,\Xi)}_{r_i}\overline{c}_{\Sigma r_{i-2}})\mathcal{E}_A+u^{r_{i-1}}\sum_{0\leq|\Xi|}\Delta^{r_{i-2},\Xi}_{r_{i-1}}\overline{c}_{\Xi r_{i-2}}]d^nx=d_H\sigma_i.$$

Using the relation (2), we obtain

$$[\sum_{0\leq|\Xi|}c^{r_i}\sum_{0\leq|\Sigma|}h^{(r_{i-2},\Sigma)(A,\Xi)}_{r_i}\overline{c}_{\Sigma r_{i-2}}d_\Xi\mathcal{E}_A+u^{r_{i-1}}\sum_{0\leq|\Xi|}\Delta^{r_{i-2},\Xi}_{r_{i-1}}\overline{c}_{\Xi r_{i-2}}]d^nx=d_H\sigma'_i.$$

The variational derivative of the both sides of this equality with respect to the antifield $\overline{c}_{r_{i-2}}$ leads to the relation

$$\sum_{0\leq|\Sigma|}\eta(h^{(r_{i-2})(A,\Xi)}_{r_i})^\Sigma d_\Sigma(c^{r_i}d_\Xi\mathcal{E}_A)+\sum_{0\leq|\Sigma|}u^{r_{i-1}}_\Sigma\eta(\Delta^{r_{i-2}}_{r_{i-1}})^\Sigma=0,$$

which takes the form

$$\sum_{0\leq|\Sigma|} d_\Sigma u^{r_{i-1}}\frac{\partial}{\partial c^{r_{i-1}}_\Sigma}u^{r_{i-2}}=\overline\delta(\alpha^{r_{i-2}}), \qquad \alpha^{r_{i-2}}=-\sum_{0\leq|\Sigma|}\eta(h^{(r_{i-2})(A,\Xi)}_{r_i})^\Sigma d_\Sigma(c^{r_i}\overline s_{\Xi A}). \tag{33}$$

Therefore, the odd graded derivations

$$u_{(k)}=u^{r_{k-1}}\frac{\partial}{\partial c^{r_{k-1}}}, \qquad u^{r_{k-1}}=\sum_{0\leq|\Lambda|} c^{r_k}_\Lambda\eta(\Delta^{r_{k-1}}_{r_k})^\Lambda, \qquad k=1,\ldots,N, \tag{34}$$

are the $k$-stage gauge supersymmetries [2]. The graded derivations $u$ (32), $u_{(k)}$ (34) are assembled into the graded derivation

$$u_e=u+\sum_{1\leq k\leq N}u_{(k)} \tag{35}$$

of ghost number 1, that we agree to call the total gauge operator. It is an ascent operator of the sequence

$$0\to\mathcal{S}^0[Q;Y]\xrightarrow{u_e}\mathcal{P}^0\{N\}_1\xrightarrow{u_e}\mathcal{P}^0\{N\}_2\xrightarrow{u_e}\cdots. \tag{36}$$

The total gauge operator (35) need not be nilpotent. One can say that gauge and higher-stage gauge supersymmetries of a Lagrangian system form an algebra on the shell if the graded derivation (35) can be extended to a graded derivation $u_E$ of ghost number 1 by means of terms of higher polynomial degree in ghosts, and $u_E$ is nilpotent on the shell. Namely, we have

$$u_E=u_e+\xi=u^A\partial_A+\sum_{1\leq k\leq N}(u^{r_{k-1}}+\xi^{r_{k-1}})\partial_{r_{k-1}}+\xi^{r_N}\partial_{r_N}, \tag{37}$$

where all the coefficients $\xi^{r_k}$, $k=0,\ldots,N$, are at least quadratic in ghosts and $(u_E\circ u_E)(f)$ is $\overline\delta$-exact for any graded function $f\in\mathcal{P}^0\{N\}\subset P^0\{N\}$. This nilpotency condition falls into a set of equalities with respect to the polynomial degree in ghosts. Let us write the first and second of them involving the coefficients $\xi^{r_{k-1}}_2$ quadratic in ghosts. We have

$$\sum_{0\leq|\Sigma|} d_\Sigma u^r\partial^\Sigma_r u^B=\overline\delta(\alpha^B_1), \qquad \sum_{0\leq|\Sigma|} d_\Sigma u^{r_{k-1}}\partial^\Sigma_{r_{k-1}}u^{r_{k-2}}=\overline\delta(\alpha^{r_{k-2}}_1), \quad 2\leq k\leq N, \tag{38}$$

$$\sum_{0\leq|\Sigma|}[d_\Sigma u^A\partial^\Sigma_A u^B+d_\Sigma\xi^r_2\partial^\Sigma_r u^B]=\overline\delta(\alpha^B_2), \tag{39}$$

$$\sum_{0\leq|\Sigma|}[d_\Sigma u^A\partial^\Sigma_A u^{r_{k-1}}+d_\Sigma\xi^{r_k}_2\partial^\Sigma_{r_k}u^{r_{k-1}}+d_\Sigma u^{r'_{k-1}}\partial^\Sigma_{r'_{k-1}}\xi^{r_{k-1}}_2]=\overline\delta(\alpha^{r_{k-1}}_2), \tag{40}$$

$$\xi^r_2=\xi^{r,\Lambda,\Sigma}_{r',r''}c^{r'}_\Lambda c^{r''}_\Sigma, \qquad \xi^{r_k}_2=\xi^{r_k,\Lambda,\Sigma}_{r,r'_k}c^r_\Lambda c^{r'_k}_\Sigma, \qquad 2\leq k\leq N. \tag{41}$$

The equalities (38) reproduce the relations (33). The equalities (39) – (40) provide the generalized commutation relations on the shell between gauge and higher-stage gauge supersymmetries, and one can think of the coefficients $\xi_2$ (41) as being *sui generis* generalized structure functions [2, 10].

## V. THE MASTER EQUATION AND EXTENDED LAGRANGIAN

The BGDA $P^*\{N\}$ (26) with the basis (27) exemplifies Lagrangian systems of the following particular type.

Let $Y_0 \to X$ be an affine bundle and $\overline{Y}_0^*$ the density-dual of the vector bundle which an affine bundle $Y_0$ is modelled on. Let $Y_1 \to X$ be a vector bundle and $\overline{Y}_1^*$ its density-dual. We consider the BGDA $\mathcal{P}^*[\overline{Y}_0^*; Y_1; Y_0; \overline{Y}_1^*]$ endowed with the basis $\{y^a, \overline{y}_a\}$, $[\overline{y}_a] = ([y^a]+1)\mathrm{mod}\,2$, whose elements $y^a$ and $\overline{y}_a$ are called the fields and antifields, respectively. Then one can associate to any Lagrangian

$$\mathfrak{L}d^nx \in \mathcal{P}^{0,n}[\overline{Y}_0^*; Y_1; Y_0; \overline{Y}_1^*] \tag{42}$$

the odd graded derivations

$$\upsilon = \overleftarrow{\mathcal{E}}^a\partial_a = \frac{\overleftarrow{\delta}\mathfrak{L}}{\delta\overline{y}_a}\frac{\partial}{\partial y^a}, \qquad \overline{\upsilon} = \overleftarrow{\partial}^a\mathcal{E}_a = \frac{\overleftarrow{\partial}}{\partial\overline{y}_a}\frac{\delta\mathfrak{L}}{\delta y^a} \tag{43}$$

of the BGDA $\mathcal{P}^*[\overline{Y}_0^*; Y_1; Y_0; \overline{Y}_1^*]$.

**Theorem 1:** The following conditions are equivalent:

(i) the graded derivation $\upsilon$ (43) is a variational supersymmetry of a Lagrangian $\mathfrak{L}d^nx$ (42),

(ii) the graded derivation $\overline{\upsilon}$ (43) is a variational supersymmetry of $\mathfrak{L}d^nx$ (42),

(iii) the composition $(\upsilon-\overline{\upsilon})\circ(\upsilon+\overline{\upsilon})$ acting on even graded functions $f \in \mathcal{P}^0[\overline{Y}_0^*; Y_1; Y_0; \overline{Y}_1^*]$ (or, equivalently, $(\upsilon+\overline{\upsilon})\circ(\upsilon-\overline{\upsilon})$ acting on the odd ones) vanishes.

*Proof:* By virtue of the formula (11), the conditions (i) and (ii) are equivalent to the equality

$$\overleftarrow{\mathcal{E}}^a\mathcal{E}_a d^nx = \frac{\overleftarrow{\delta}\mathfrak{L}}{\delta\overline{y}_a}\frac{\delta\mathfrak{L}}{\delta y^a}d^nx = d_H\sigma, \tag{44}$$

called the master equation. The equality (44), in turn, is equivalent to the condition that the odd graded density $\overleftarrow{\mathcal{E}}^a\mathcal{E}_a d^nx$ is variationally trivial. Replacing the right variational derivatives $\overleftarrow{\mathcal{E}}^a$ in the equality (44) with the left ones $(-1)^{[a]+1}\mathcal{E}^a$. We obtain

$$\sum_a (-1)^{[a]}\mathcal{E}^a\mathcal{E}_a d^nx = d_H\sigma.$$

The variational operator acting on this relation leads to the equalities

$$\sum_{0\leq|\Lambda|}(-1)^{[a]+|\Lambda|}d_\Lambda(\partial_b^\Lambda(\mathcal{E}^a\mathcal{E}_a)) = \sum_{0\leq|\Lambda|}(-1)^{[a]}[\eta(\partial_b\mathcal{E}^a)^\Lambda\mathcal{E}_{\Lambda a}+\eta(\partial_b\mathcal{E}_a)^\Lambda\mathcal{E}^a_\Lambda)]=0,$$
$$\sum_{0\leq|\Lambda|}(-1)^{[a]+|\Lambda|}d_\Lambda(\partial^{\Lambda b}(\mathcal{E}^a\mathcal{E}_a)) = \sum_{0\leq|\Lambda|}(-1)^{[a]}[\eta(\partial^b\mathcal{E}^a)^\Lambda\mathcal{E}_{\Lambda a}+\eta(\partial^b\mathcal{E}_a)\mathcal{E}^a_\Lambda]=0.$$

Due to the formulas (10), these equalities are brought into the form

$$\sum_{0\leq|\Lambda|}(-1)^{[a]}[(-1)^{[b]([a]+1)}\partial^{\Lambda a}\mathcal{E}_b\mathcal{E}_{\Lambda a}+(-1)^{[b][a]}\partial^\Lambda_a\mathcal{E}_b\mathcal{E}^a_\Lambda]=0, \tag{45}$$
$$\sum_{0\leq|\Lambda|}(-1)^{[a]}[(-1)^{([b]+1)([a]+1)}\partial^{\Lambda a}\mathcal{E}^b\mathcal{E}_{\Lambda a}+(-1)^{([b]+1)[a]}\partial^\Lambda_a\mathcal{E}^b\mathcal{E}^a_\Lambda]=0 \tag{46}$$

for all $\mathcal{E}_b$ and $\mathcal{E}^b$. Returning to the right variational derivatives, we obtain the relations

$$\overleftarrow{\partial}^{\Lambda a}(\mathcal{E}_b)\mathcal{E}_{\Lambda a}+(-1)^{[b]}\overleftarrow{\mathcal{E}}{}^a_\Lambda\partial^\Lambda_a\mathcal{E}_b=0, \qquad \overleftarrow{\mathcal{E}}{}^a_\Lambda\partial^\Lambda_a\overleftarrow{\mathcal{E}}{}^b+(-1)^{[b]+1}\overleftarrow{\partial}^{\Lambda a}(\overleftarrow{\mathcal{E}}{}^b)\mathcal{E}_{\Lambda a}=0. \tag{47}$$

A direct computation shows that they are equivalent to the condition (iii).

It is readily observed that the equalities (45) – (46) and, equivalently, the equalities (47) are Noether identities of a Lagrangian (42).

Note that any variationally trivial Lagrangian satisfies the master equation. We say that a solution of the master equation is not trivial if both the graded derivations (43) are not zero. It is readily observed that, if a Lagrangian $\mathfrak{L}d^nx$ provides a nontrivial solution of the master equation and $L_0$ is a variationally trivial Lagrangian, the sum $\mathfrak{L}d^nx+L_0$ is also a nontrivial solution of the master equation.

Let us return to an original graded Lagrangian system $(\mathcal{P}^*[Q;Y],L)$ and its extension $(P^*\{N\},L_e,u_e)$ to ghosts and antifields, together with the odd graded derivations (43) which read

$$\upsilon_e=\vartheta+\vartheta_e=\frac{\overleftarrow{\delta}\mathcal{L}_1}{\delta\overline{s}_A}\frac{\partial}{\partial s^A}+\sum_{0\leq k\leq N}\frac{\overleftarrow{\delta}\mathcal{L}_1}{\delta\overline{c}_{r_k}}\frac{\partial}{\partial c^{r_k}},$$
$$\overline{\upsilon}_e=\overline{\vartheta}+\delta_N=\frac{\overleftarrow{\partial}}{\partial\overline{s}_A}\frac{\delta\mathcal{L}_1}{\delta s^A}+[\frac{\overleftarrow{\partial}}{\partial\overline{s}_A}\frac{\delta\mathcal{L}}{\delta s^A}+\sum_{0\leq k\leq N}\frac{\overleftarrow{\partial}}{\partial\overline{c}_{r_k}}\frac{\delta\mathcal{L}_1}{\delta c^{r_k}}].$$

An original Lagrangian provides a trivial solution of the master equation. A problem is to extend the Lagrangian $L_e$ (28) to a solution of the master equation

$$L_e+L'=L+L_1+L_2+\cdots \tag{48}$$

by means of even terms $L_i$ of zero antifield number and polynomial degree $i>1$ in ghosts.

**Theorem 2:** The Lagrangian $L_e$ (28) can be extended to a solution (48) of the master equation only if the total gauge operator $u_e$ (35) is extended to a graded derivation, nilpotent on the shell. This extension is independent of antifields.

*Proof:* Given a Lagrangian (48), the corresponding graded derivations (43) read

$$\upsilon = \frac{\overleftarrow{\delta}(\mathcal{L}_1 + \mathcal{L}')}{\delta \overline{s}_A}\frac{\partial}{\partial s^A} + \sum_{0\leq k\leq N} \frac{\overleftarrow{\delta}(\mathcal{L}_1 + \mathcal{L}')}{\delta \overline{c}_{r_k}}\frac{\partial}{\partial c^{r_k}}, \tag{49}$$

$$\overline{\upsilon} = \frac{\overleftarrow{\partial}}{\partial \overline{s}_A}\frac{\delta(\mathcal{L} + \mathcal{L}_1 + \mathcal{L}')}{\delta s^A} + \sum_{0\leq k\leq N} \frac{\overleftarrow{\partial}}{\partial \overline{c}_{r_k}}\frac{\delta(\mathcal{L}_1 + \mathcal{L}')}{\delta c^{r_k}}. \tag{50}$$

Then the condition (iii) of Theorem 1 can be written in the form (6) as

$$(\upsilon + \overline{\upsilon})(\upsilon) = 0, \qquad (\upsilon + \overline{\upsilon})(\overline{\upsilon}) = 0. \tag{51}$$

It falls into a set of equalities with respect to the polynomial degree in antifields. Let us put

$$\upsilon = \upsilon^0 + \upsilon^1 + \upsilon', \qquad \overline{\upsilon} = \overline{\upsilon}^0 + \overline{\upsilon}',$$

where $\upsilon^0$ and $\upsilon^1$ are the parts of $\upsilon$ (49) of zero and first polynomial degree in antifields, respectively, and $\overline{\upsilon}^0$ is that of $\overline{\upsilon}$ (50) independent of antifields. It is readily observed that $\overline{\upsilon}^0 = \overline{\delta}$ is the Koszul–Tate differential. Let us consider the part of the first equality (51) which is independent of antifields. It reads

$$\upsilon^0(\upsilon^0) + \overline{\upsilon}^0(\upsilon^1) = \upsilon^0(\upsilon^0) + \overline{\delta}(\upsilon^1) = 0, \tag{52}$$

i.e., the graded derivation $\upsilon^0$ is nilpotent on the shell. Since the part of $\upsilon^0$ linear in ghosts is exactly the total gauge operator $u_e$ (35), the graded derivation $\upsilon^0$ provides a desired extension $u_E = \upsilon^0$ (37) of $u_e$ which is nilpotent on the shell.

Theorem 2 shows that the Lagrangian $L_e$ (28) is extended to a solution of the master equation only if the gauge and higher-stage gauge supersymmetries of an original Lagrangian $L$ form an algebra on the shell.

In order to formulate the sufficient condition, let us assume that the gauge and higher-stage gauge supersymmetries of an original Lagrangian $L$ form an algebra, i.e., we have a nilpotent extension of the total gauge operator $u_e$ (35).

**Theorem 3:** Let the total gauge operator $u_e$ (35) admit a nilpotent extension $u_E$ (37) independent of antifields. Then the extended Lagrangian

$$L_E = L_e + \sum_{0\leq k\leq N} \xi^{r_k}\overline{c}_{r_k} d^n x \tag{53}$$

satisfies the master equation.

*Proof:* If the graded derivation $\upsilon^0 = u_E$ is nilpotent, then $\overline{\delta}(\upsilon^1) = 0$ by virtue of the equation (52). It follows that the part $L_1^2$ of the Lagrangian $L_e$ quadratic in antifields obeys the relations $\overline{\delta}(\overleftarrow{\delta}^{r_k}(\mathcal{L}_1^2)) = 0$ for all indices $r_k$. This part consists of the terms $h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}$ (22), which consequently are $\overline{\delta}$-closed. Then the summand $G_{r_k}$ of each cocycle $\Delta_{r_k}$ (22) is $\delta_{k-1}$-closed in accordance with the relation (24). It follows that its summand $h_{r_k}$ is also $\delta_{k-1}$-closed and, consequently, $\delta_{k-2}$-closed. Hence it is $\delta_{k-1}$-exact by virtue of the homology regularity condition. Therefore, $\Delta_{r_k}$ is reduced only to the summand $G_{r_k}$ linear in antifields. It follows that the Lagrangian $L_1$ (28) is linear in antifields. In this case, we have $u^A = \overleftarrow{\delta}^A(\mathcal{L}_e)$, $u^{r_k} = \overleftarrow{\delta}^{r_k}(\mathcal{L}_e)$ for all indices $A$ and $r_k$ and, consequently,

$$\upsilon^A = \overleftarrow{\delta}^A(\mathcal{L}_E), \qquad \upsilon^{r_k} = \overleftarrow{\delta}^{r_k}(\mathcal{L}_E),$$

i.e., $\upsilon^0 = \upsilon$ is the graded derivation (43) defined by the Lagrangian (53). Then the nilpotency condition $\upsilon^0(\upsilon^0) = 0$ takes the form

$$\upsilon^0(\overleftarrow{\delta}^A(\mathcal{L}_E)) = 0, \qquad \upsilon^0(\overleftarrow{\delta}^{r_k}(\mathcal{L}_E)) = 0.$$

Hence, we obtain

$$\upsilon^0(L_E) = u(L) + \upsilon^0[\overleftarrow{\delta}^A(\mathcal{L}_E)\overline{s}_A + \sum_{0\leq k\leq N} \overleftarrow{\delta}^{r_k}(\mathcal{L}_E)\overline{c}_{r_k}]d^nx = d_H\sigma,$$

i.e., $\upsilon^0 = u_E$ is a variational supersymmetry of the extended Lagrangian (53). Thus, this Lagrangian satisfies the master equation in accordance with Theorem 1.

## VI. THE GAUGE-FIXING LAGRANGIAN

Let us further restrict our consideration to a Lagrangian system which satisfies the condition of Theorem 3, i.e., its gauge and higher-stage gauge supersymmetries form an algebra. Its total gauge operator

$$u_e = u + \sum_{1\leq k\leq N} u_{(k)},$$
$$u = u^A\frac{\partial}{\partial s^A}, \qquad u^A = \sum_{0\leq|\Lambda|} c_\Lambda^r \eta(\Delta_r^A)^\Lambda,$$
$$u_{(k)} = u^{r_{k-1}}\frac{\partial}{\partial c^{r_{k-1}}}, \qquad u^{r_{k-1}} = \sum_{0\leq|\Lambda|} c_\Lambda^{r_k}\eta(\Delta_{r_k}^{r_{k-1}})^\Lambda, \qquad k=1,\ldots,N,$$

admits a nilpotent extension

$$u_E = u_e + \xi = u^A\partial_A + \sum_{1\leq k\leq N} (u^{r_{k-1}} + \xi^{r_{k-1}})\partial_{r_{k-1}} + \xi^{r_N}\partial_{r_N}, \tag{54}$$

called the BRST operator. Accordingly, an original Lagrangian $L$ is extended to the Lagrangian

$$L_E = L + [u^A \overline{s}_A + \sum_{1\leq k\leq N} (u^{r_{k-1}} + \xi^{r_{k-1}})\overline{c}_{r_{k-1}} + \xi^{r_N}\overline{c}_{r_N}]d^n x, \tag{55}$$

which differs from the Lagrangian (53) in a $d_H$-exact term. The extended Lagrangian (55) obeys the master equation

$$\frac{\overleftarrow{\delta}\mathcal{L}_E}{\delta \overline{s}_A}\frac{\delta \mathcal{L}_E}{\delta s^A}d^n x + \sum_{0\leq k\leq N}\frac{\overleftarrow{\delta}\mathcal{L}_E}{\delta \overline{c}_{r_k}}\frac{\delta \mathcal{L}_E}{\delta c^{r_k}}d^n x = d_H\sigma$$

and the relation

$$u_E = \frac{\overleftarrow{\delta}\mathcal{L}_E}{\delta \overline{s}_A}\frac{\partial}{\partial s^A} + \sum_{0\leq k\leq N}\frac{\overleftarrow{\delta}\mathcal{L}_E}{\delta \overline{c}_{r_k}}\frac{\partial}{\partial c^{r_k}}.$$

The Noether identities (47) show that the extended Lagrangian $L_E$ (55) is degenerate. Following the BV quantization procedure, we aim to replace the antifields in $L_E$ with gauge-fixing terms in order to obtain a non-degenerate Lagrangian [4, 5].

For this purpose, let us consider an odd graded density $\Psi d^n x$ of antifield number 1 which depends on original fields $s^A$ and ghosts $c^{r_k}$, $k = 0, \ldots, N$, but not antifields $\overline{s}_A$, $\overline{c}_{r_k}$, $k = 0, \ldots, N$. In order to satisfy these conditions, new field variables must be introduced because all the ghosts are of negative antifield numbers. Therefore, let us enlarge the BGDA $P^*\{N\}$ (26) to the BGDA

$$\overline{P}^*\{N\} = \mathcal{P}^*[\overline{E}^*_N \cdots \overline{E}^* \overline{V}^*_N \cdots \overline{V}^* V_N \cdots V \overline{Y}^*; Q; Y; \overline{Q}^* E \cdots E_N \overline{E}^* \cdots \overline{E}^*_N \overline{V}^* \cdots \overline{V}^*_N],$$

possessing the basis

$$\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}, c^*_r, c^*_{r_1}, \ldots, c^*_{r_N}, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}\},$$

where $[c^*_{r_k}] = [c^{r_k}]$ and $\mathrm{Ant}[c^*_{r_k}] = k + 1$, $k = 0, \ldots, N$. We agree to call $c^*_r, c^*_{r_1}, \ldots, c^*_{r_N}$ the antighosts. Then, we choose $\Psi d^n x$ as an element of $\overline{P}^{0,n}\{N\}$. It is traditionally called the gauge-fixing fermion.

Let us replace all the antifields in the Lagrangian $L_E$ (55) with the gauge fixing terms

$$\overline{s}_A = \frac{\delta\Psi}{\delta s^A}, \qquad \overline{c}_{r_k} = \frac{\delta\Psi}{\delta c^{r_k}}, \qquad k = 0, \ldots, N.$$

We obtain the Lagrangian

$$L_\Psi = L + [u^A_E \frac{\delta\Psi}{\delta s^A} + \sum_{0\leq k\leq N} u^{r_k}_E \frac{\delta\Psi}{\delta c^{r_k}}]d^n x = L + u_E(\Psi)d^n x + d_H\sigma, \tag{56}$$

which is an element of the BGDA

$$\mathfrak{P}^*\{N\} = \mathcal{P}^*[\overline{V}^*_N \cdots \overline{V}^* V_N \cdots V; Q; Y; E \cdots E_N \overline{E}^* \cdots \overline{E}^*_N] \subset \overline{P}^*\{N\}, \tag{57}$$

possessing the basis

$$\{s^A, c^r, c^{r_1}, \ldots, c^{r_N}, c^*_r, c^*_{r_1}, \ldots, c^*_{r_N}\}. \tag{58}$$

The BRST operator $u_E$ (54) is obviously a graded derivation of the BGDA $\mathfrak{P}^*\{N\}$. A glance at the equalities

$$u_E(L_\Psi) = u(L) + u_E(u_E(\Psi)d^n x + d_H\sigma)d^n x = d_H\sigma'$$

shows that $u_E$ is a variational supersymmetry of the Lagrangian $L_\Psi$ (56). It however is not a gauge supersymmetry of $L_\Psi$ if $L_\Psi$ depends on all the ghosts $c^{r_k}$, $k = 0, \ldots, N$, i.e., no ghost is a gauge parameter. Therefore, we require that

$$\frac{\delta\Psi}{\delta s^A} \neq 0, \qquad \frac{\delta\Psi}{\delta c^{r_k}} \neq 0, \qquad k = 0, \ldots, N-1. \tag{59}$$

In this case, Noether identities for the Lagrangian $L_\Psi$ (56) come neither from the BRST symmetry $u_E$ nor the equalities (47). One also put

$$\Psi = \sum_{0\leq k\leq N} \Psi^{r_k} c^*_{r_k}. \tag{60}$$

Finally, let $h^{r_k r'_k}$ be a non-degenerate bilinear form $h^{r_k r'_k}$ for each $k = 0, \ldots, N-1$ whose coefficients are either real numbers or functions on $X$. Then a desired gauge-fixing Lagrangian is written in the form

$$\begin{aligned} L_{GF} = L_\Psi + \sum_{0\leq k\leq N} \frac{h_{r_k r'_k}}{2} \Psi^{r_k} \Psi^{r'_k} d^n x = \\ L + [u^A_E \frac{\delta\Psi}{\delta s^A} + \sum_{0\leq k\leq N} u^{r_k}_E \frac{\delta\Psi}{\delta c^{r_k}}]d^n x + \sum_{0\leq k\leq N} \frac{h_{r_k r'_k}}{2} \Psi^{r_k} \Psi^{r'_k} d^n x = \\ L + \sum_{0\leq k\leq N} u_E(\Psi^{r_k}) c^*_{r_k} d^n x + \sum_{0\leq k\leq N} \frac{h_{r_k r'_k}}{2} \Psi^{r_k} \Psi^{r'_k} d^n x + d_H\sigma. \end{aligned} \tag{61}$$

The BRST operator $u_E$ (54) fails to be a variational symmetry of the Lagrangian (61). However, it can be extended to the graded derivation

$$\widehat{u} = u_E - \sum_{0\leq k\leq N} \frac{\overleftarrow{\partial}}{\partial c^*_{r_k}} h_{r_k r'_k} \Psi^{r'_k} \tag{62}$$

of the BGDA $\mathfrak{P}^*\{N\}$ (57) which is easily proved to be a variational supersymmetry of the gauge-fixing Lagrangian (61). We agree to call $\widehat{u}$ (62) the gauge-fixing BRST symmetry though it is not nilpotent.

Of course, the Lagrangian $L_{GF}$ essentially depends on a choice of the gauge-fixing fermion $\Psi$ which must satisfy the conditions (59) and (60). These conditions need not guarantee that that the Lagrangian $L_{GF}$ is non-degenerate, but we assume that this is well.

## VII. QUANTIZATION

Let us quantize a non-degenerate Lagrangian system $(\mathfrak{P}^*\{N\}, L_{GF})$. Though our results lie in the framework of perturbed QFT, we start with algebraic QFT.

In algebraic QFT, a quantum field system is characterized by a topological $^*$-algebra $A$ and a continuous positive form $f$ on $A$ [11, 12]. For the sake of simplicity, let us consider even scalar fields on the Minkowski space $X = \mathbb{R}^n$. One associates to them the Borchers algebra $A_\Phi$ of tensor products of the Schwartz space $\Phi = S(\mathbb{R}^n)$ of smooth complex functions of rapid decreasing at infinity on $\mathbb{R}^n$. These are complex smooth functions $f$ such that the quantities

$$|\phi|_{k,m} = \max_{|\alpha|\leq k} \sup_x (1+x^2)^m \left|\frac{\partial^{|\alpha|}\phi}{\partial^{\alpha_1}x^1\cdots\partial^{\alpha_n}x^n}\right|, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n, \tag{63}$$

are finite for all $k, m \in \mathbb{N}$ for $n$-tuples of natural numbers $\alpha = (\alpha_1, \ldots, \alpha_n)$. The space $S(\mathbb{R}^n)$ is nuclear with respect to the topology determined by the seminorms (63). Its topological dual is the space $S'(\mathbb{R}^n)$ of tempered distributions [13, 14]. The corresponding contraction form is written as

$$\langle \phi, \psi\rangle = \int \phi(x)\phi(x) d^n x, \qquad \phi \in S(\mathbb{R}^n), \qquad \psi \in S'(\mathbb{R}^n).$$

The space $S(\mathbb{R}^n)$ is provided with the non-degenerate separately continuous Hermitian form

$$\langle \phi|\phi'\rangle = \int \phi(x)\overline{\phi}'(x) d^n x.$$

The completion of $S(\mathbb{R}^n)$ with respect to this form is the space $L^2_C(\mathbb{R}^n)$ of square integrable complex functions on $\mathbb{R}^n$. We have the rigged Hilbert space

$$S(\mathbb{R}^n) \subset L^2_C(\mathbb{R}^n) \subset S'(\mathbb{R}^n).$$

Let $\mathbb{R}_n$ denote the dual of $\mathbb{R}^n$ coordinated by $(p_\lambda)$. The Fourier transform

$$\phi^F(p) = \int \phi(x) e^{ipx} d^n x, \qquad px = p_\lambda x^\lambda, \tag{64}$$

$$\phi(x) = \int \phi^F(p) e^{-ipx} d_n p, \qquad d_n p = (2\pi)^{-n} d^n p, \tag{65}$$

provides an isomorphism between the spaces $S(\mathbb{R}^n)$ and $S(\mathbb{R}_n)$. The Fourier transform of distributions is defined by the condition

$$\int \psi(x)\phi(x)d^n x = \int \psi^F(p)\phi^F(-p)d_n p,$$

and is written in the form (64) – (65). It provides an isomorphism between the spaces of distributions $S'(\mathbb{R}^n)$ and $S'(\mathbb{R}_n)$.

Since $\overset{n}{\otimes} S(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^{nk})$, a state $f$ of the Borchers algebra $A_\Phi$ is represented by distributions

$$f_k(\phi_1 \cdots \phi_k) = \int W_k(x_1, \dots, x_k)\phi_1(x_1)\cdots\phi_k(x_k)d^n x_1 \dots d^n x_k, \qquad W_k \in S'(\mathbb{R}^{nk}).$$

In particular, the $k$-point Wightman functions $W_k$ describe free fields in the Minkowski space. The complete Green functions characterize quantum fields created at some instant and annihilated at another one. They are given by the chronological functionals

$$f^c(\phi_1 \cdots \phi_k) = \int W^c_k(x_1, \dots, x_k)\phi_1(x_1)\cdots\phi_k(x_k)d^n x_1 \dots d^n x_k, \tag{66}$$

$$W^c_k(x_1, \dots, x_k) = \sum_{(i_1 \dots i_k)} \theta(x^0_{i_1} - x^0_{i_2})\cdots\theta(x^0_{i_{k-1}} - x^0_{i_k})W_k(x_1, \dots, x_k), \quad W_{nk} \in S'(\mathbb{R}^{nk}),$$

where $\theta$ is the step function, and the sum runs through all permutations $(i_1 \dots i_k)$ of the numbers $1, \dots, k$. However, the chronological functionals (66) need not be continuous and positive. At the same time, they issue from the Wick rotation of Euclidean states of the Borchers algebra $A_\Phi$ describing quantum fields in an interaction zone [15, 16]. Since the chronological functionals (66) are symmetric, these Euclidean states are states of the corresponding commutative tensor algebra $B_\Phi$. This is the enveloping algebra of the Lie algebra of the group $T(\Phi)$ of translations in $\Phi$. Therefore one can obtain a state of $B_\Phi$ as a vector form of a strong-continuous unitary cyclic representation of $T(\Phi)$ [17]. Such a representation is characterized by a positive-definite continuous generating function $Z$ on $\Phi$. By virtue of the Bochner theorem [17], this function is the Fourier transform

$$Z(\phi) = \int_{\Phi'} \exp[i\langle\phi, w\rangle]d\mu(w) \tag{67}$$

of a positive measure $\mu$ of total mass 1 on the topological dual $\Phi'$ of $\Phi$. If the function $\alpha \to Z(\alpha\phi)$ on $\mathbb{R}$ is analytic at 0 for each $\phi \in \Phi$, a state $F$ of $B_\Phi$ is given by the expression

$$F_k(\phi_1 \cdots \phi_k) = i^{-k}\frac{\partial}{\partial\alpha^1}\cdots\frac{\partial}{\partial\alpha^k}Z(\alpha^i\phi_i)|_{\alpha^i=0} = \int \langle\phi_1, w\rangle\cdots\langle\phi_k, w\rangle d\mu(w). \tag{68}$$

Then one can regard $Z$ (67) as a generating functional of Euclidean Green functions $F_k$ (68). A problem is that, if a field Lagrangian is a polynomial of degree more than two, a

generating functional $Z$ (67), Green functions and, consequently, Ward identities fail to be written in an explicit form.

Therefore, let us quantize the above mentioned non-degenerate Lagrangian system $(\mathfrak{P}^*\{N\}, L_{GF})$ in the framework of perturbed QFT. We assume that $L_{GF}$ is a Lagrangian of Euclidean fields on $X = \mathbb{R}^n$, coordinated by $(x^\lambda)$. The key point is that an algebra of Euclidean quantum fields is graded commutative, and there are homomorphisms of the graded commutative ring $\mathfrak{P}^0\{N\}$ of classical fields to this algebra. These monomorphisms enable one to define the BRST operator $\widehat{u}$ (62) on Euclidean quantum fields.

Let $\mathcal{Q}$ be the graded complex vector space whose basis is the basis (58) for the BGDA $\mathfrak{P}^*\{N\}$. The common symbols $q^a$ further stand for elements of this basis. Let us consider the tensor product

$$\Phi = \mathcal{Q} \otimes S'(\mathbb{R}^n) \tag{69}$$

of the graded vector space $\mathcal{Q}$ and the space $S'(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n$. One can think of elements $\Phi$ (69) as being $\mathcal{Q}$-valued distributions on $\mathbb{R}^n$. Let us consider the subspace $T(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ of functions $\exp\{ipx'\}$, $p \in \mathbb{R}_n$, which are generalized eigenvectors of translations in $\mathbb{R}^n$ acting on $S(\mathbb{R}^n)$. Let us denote $f_p^a = q^a \otimes \exp\{ipx'\}$. Then any element $\phi$ of $\Phi$ can be written in the form

$$\phi(x') = q^a \otimes \phi_a(x') = \int \phi_a(p)\phi_p^a d_n p, \tag{70}$$

where $\phi_a(p) \in S'(\mathbb{R}_n)$ are the Fourier transforms of $\phi_a(-x')$. For instance, we have the $\mathcal{Q}$-valued distributions

$$\phi_x^a(x') = \int \phi_p^a e^{-ipx} d_n p = q^a \otimes \delta(x - x'), \tag{71}$$

$$\phi_{x\Lambda}^a(x') = \int (-i)^k p_{\lambda_1} \cdots p_{\lambda_k} \phi_p^a e^{-ipx} d_n p. \tag{72}$$

In the framework of perturbed Euclidean QFT, we associate to a non-degenerate Lagrangian system $(\mathfrak{P}^*\{N\}, L_{GF})$ the graded commutative tensor algebra $B_\Phi$ generated by elements of the graded vector space $\Phi$ (69) and the following state $\langle . \rangle$ of $B_\Phi$. For any $x \in X$, there is a homomorphism

$$\gamma_x : f_{a_1 \ldots a_r}^{\Lambda_1 \ldots \Lambda_r} q_{\Lambda_1}^{a_1} \cdots q_{\Lambda_r}^{a_r} \mapsto f_{a_1 \ldots a_r}^{\Lambda_1 \ldots \Lambda_r}(x) \phi_{x\Lambda_1}^{a_1} \cdots \phi_{x\Lambda_r}^{a_r}, \qquad f_{a_1 \ldots a_r}^{\Lambda_1 \ldots \Lambda_r} \in C^\infty(X), \tag{73}$$

of the $C^\infty(X)$-ring $\mathfrak{P}^0\{N\}$ to the algebra $B_\Phi$ which sends elements $q_\Lambda^a \in \mathfrak{P}^0\{N\}$ to the elements $\phi_{x\Lambda}^a \in B_\Phi$ (72), and replaces coefficient functions $f$ of elements of $\mathfrak{P}^0\{N\}$ with their values $f(x)$ at a point $x$. It should be emphasized that $f_{a_1 \ldots a_r}^{\Lambda_1 \ldots \Lambda_r}(x) \phi_{x\Lambda_1}^{a_1} \cdots \phi_{x\Lambda_r}^{a_r}$ in the expression (73) is the graded commutative tensor product of distributions, but not their

product which is ill defined. Then the above mentioned state $\langle . \rangle$ of $B_\Phi$ is given by symbolic functional integrals

$$\langle \phi_1 \cdots \phi_k \rangle = \frac{1}{\mathcal{N}} \int \phi_1 \cdots \phi_k \exp\{-\int \mathcal{L}_{GF}(\phi^a_p) d^n x\} \prod_p [d\phi^a_p], \tag{74}$$

$$\mathcal{N} = \int \exp\{-\int \mathcal{L}_{GF}(\phi^a_p) d^n x\} \prod_p [d\phi^a_p], \tag{75}$$

$$\mathcal{L}_{GF}(\phi^a_p) = \mathcal{L}_{GF}(\phi^a_{x\Lambda}) = \mathcal{L}_{GF}(x, \gamma_x(q^a_\Lambda)), \tag{76}$$

where $\phi_i$ and $\gamma_x(q^a_\Lambda) = \phi^a_{x\Lambda}$ are given by the formulas (70) and (72), respectively. The forms (74) are expressed into the forms

$$\langle \phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k} \rangle = \frac{1}{\mathcal{N}} \int \phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k} \exp\{-\int \mathcal{L}_{GF}(\phi^a_p) d^n x\} \prod_p [d\phi^a_p], \tag{77}$$

$$\langle \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \rangle = \frac{1}{\mathcal{N}} \int \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \exp\{-\int \mathcal{L}_{GF}(\phi^a_{x\Lambda}) d^n x\} \prod_x [d\phi^a_x], \tag{78}$$

which provide Euclidean Green functions. It should be emphasized that, in contrast with a measure $\mu$ in the expression (67), the term $\prod\limits_p [d\phi^a_p]$ in the formulas (74) – (75) fail to be a true measure on $T(\mathbb{R}^n)$ because the Lebesgue measure on infinite-dimensional vector spaces need not exist. Nevertheless, treated like integrals over a finite-dimensional vector space, the functional integrals (77) - (78) restart Euclidean Green functions in the Feynman diagram technique. Certainly, these Green functions are singular, unless regularization and renormalization techniques are involved.

## VIII. WARD IDENTITIES

Since the graded derivation (62)

$$\widehat{u} = \sum_{0 \leq |\Lambda|} \widehat{u}^a_\Lambda(q^b_\Sigma) \partial^\Lambda_a$$

of the ring $\mathfrak{P}^0\{N\}$ is a $C^\infty(X)$-linear morphism over $\mathrm{Id}\, X$, it induces the graded derivation

$$\widehat{u}_x = \gamma_x \circ \widehat{u} \circ \gamma_x^{-1} : \phi^a_{x\Lambda} \to (x, q^a_\Lambda)) \to \widehat{u}^a_\Lambda(x, q^b_\Sigma) \to \widehat{u}^a_\Lambda(x, \gamma_x(q^b_\Sigma)) = \widehat{u}^a_{x\Lambda}(\phi^b_{x\Sigma}) \tag{79}$$

of the range $\gamma_x(\mathfrak{P}^0\{N\}) \subset B_\Phi$ of the homomorphism $\gamma_x$ (73) for each $x \in X = \mathbb{R}^n$. The maps $\widehat{u}_x$ (79) yield the maps

$$\widehat{u}_p : \phi^a_p = \int \phi^a_x e^{ipx} d^n x \to \int \widehat{u}_x(\phi^a_x) e^{ipx} d^n x = \int \widehat{u}^a_x(\phi^b_{x\Sigma}) e^{ipx} d^n x =$$
$$\int \widehat{u}^a_x(\int (-i)^k p'_{\sigma_1} \cdots p'_{\sigma_k} \phi^b_{p'} e^{-ip'x} d_n p') e^{ipx} d^n x = \widehat{u}^a_p, \qquad p \in \mathbb{R}_n,$$

and, as a consequence, the graded derivation

$$\widehat{u}(\phi) = \int \phi_a(p)\widehat{u}(\phi^a_p)d_np = \int \phi_a(p)\widehat{u}^a_p d_np$$

of the algebra $B_\Phi$. We agree to call it quantum BRST transformation. It can be written in the symbolic form

$$\widehat{u} = \int u^a_p \frac{\partial}{\partial\phi^a_p} d_np, \qquad \frac{\partial\phi^b_{p'}}{\partial\phi^a_p} = \delta^b_a\delta(p'-p), \tag{80}$$

$$\widehat{u} = \int u^a_x \frac{\partial}{\partial\phi^a_x} d^nx, \qquad \frac{\partial\phi^b_{x'\Lambda}}{\partial\phi^a_x} = \delta^b_a \frac{\partial}{\partial x'^{\lambda_1}} \cdots \frac{\partial}{\partial x'^{\lambda_k}}\delta(x'-x). \tag{81}$$

Let $\alpha$ be an odd element. We consider the automorphism

$$\widehat{U} = \exp\{\alpha\widehat{u}\} = \mathrm{Id} + \alpha\widehat{u}$$

of the algebra $B_\Phi$. This automorphism yields a new state $\langle . \rangle'$ of $B_\Phi$ given by the relations

$$\begin{aligned}
&\langle\phi_1\cdots\phi_k\rangle = \langle\widehat{U}(\phi_1)\cdots\widehat{U}(\phi_k)\rangle' = \\
&\qquad \frac{1}{\mathcal{N}'}\int \widehat{U}(\phi_1)\cdots\widehat{U}(\phi_k)\exp\{-\int \mathcal{L}_{GF}(\widehat{U}(\phi^a_p))d^nx\}\prod_p[d\widehat{U}(\phi^a_p)], \\
&\mathcal{N}' = \int \exp\{-\int \mathcal{L}_{GF}(\widehat{U}(\phi^a_p))d^nx\}\prod_p[d\widehat{U}(\phi^a_p)].
\end{aligned}$$

Let us apply these relations to the Green functions (77) – (78).

Since the graded derivation $\widehat{u}$ (62) is a variational supersymmetry of the Lagrangian $L_{GF}$ (57), we obtain from the relations (76) that

$$\begin{aligned}
\int \mathcal{L}_{GF}(\widehat{U}(\phi^a_{x\Lambda}))d^nx &= \int \mathcal{L}_{GF}(\phi^a_{x\Lambda})d^nx, \\
\int \mathcal{L}_{GF}(\widehat{U}(\phi^a_p))d^nx &= \int \mathcal{L}_{GF}(\phi^a_p)d^nx.
\end{aligned}$$

It is a property of symbolic functional integrals that

$$\begin{aligned}
\prod_p[d\widehat{U}(\phi^a_p)] &= (1+\alpha\int \frac{\partial\widehat{u}^a_p}{\partial\phi^a_p}d_np)\prod_p[d\phi^a_p] = (1+\alpha\mathrm{Sp}(\widehat{u}))\prod_p[d\phi^a_p], \\
\prod_x[d\widehat{U}(\phi^a_x)] &= (1+\alpha\int \frac{\partial\widehat{u}^a_x}{\partial\phi^a_x}d^nx)\prod_x[d\phi^a_x] = (1+\alpha\mathrm{Sp}(\widehat{u}))\prod_x[d\phi^a_x].
\end{aligned}$$

Then the desired Ward identities for the Green functions (77) – (78) read

$$\langle\widehat{u}(\phi^{a_1}_{p_1}\cdots\phi^{a_k}_{p_k})\rangle + \langle\phi^{a_1}_{p_1}\cdots\phi^{a_k}_{p_k}\mathrm{Sp}(\widehat{u})\rangle - \langle\phi^{a_1}_{p_1}\cdots\phi^{a_k}_{p_k}\rangle\langle\mathrm{Sp}(\widehat{u})\rangle = 0, \tag{82}$$

$$\sum_{i=1}^{k}(-1)^{[a_1]+\cdots+[a_{i-1}]}\langle\phi^{a_1}_{p_1}\cdots\phi^{a_{i-1}}_{p_{i-1}}\widehat u^{a_i}_{p_i}\phi^{a_{i+1}}_{p_{i+1}}\cdots\phi^{a_k}_{p_k}\rangle +$$
$$\langle\phi^{a_1}_{p_1}\cdots\phi^{a_k}_{p_k}\int\frac{\partial\widehat u^a_p}{\partial\phi^a_p}d_np\rangle-\langle\phi^{a_1}_{p_1}\cdots\phi^{a_k}_{p_k}\rangle\langle\int\frac{\partial\widehat u^a_p}{\partial\phi^a_p}d_np\rangle=0,$$
$$\langle\widehat u(\phi^{a_1}_{x_1}\cdots\phi^{a_k}_{x_k})\rangle+\langle\phi^{a_1}_{x_1}\cdots\phi^{a_k}_{x_k}\mathrm{Sp}(\widehat u)\rangle-\langle\phi^{a_1}_{x_1}\cdots\phi^{a_k}_{x_k}\rangle\langle\mathrm{Sp}(\widehat u)\rangle=0, \tag{83}$$
$$\sum_{i=1}^{k}(-1)^{[a_1]+\cdots+[a_{i-1}]}\langle\phi^{a_1}_{x_1}\cdots\phi^{a_{i-1}}_{x_{i-1}}\widehat u^{a_i}_{x_i}\phi^{a_{i+1}}_{x_{i+1}}\cdots\phi^{a_k}_{x_k}\rangle +$$
$$\langle\phi^{a_1}_{x_1}\cdots\phi^{a_k}_{x_k}\int\frac{\partial\widehat u^a_x}{\partial\phi^a_x}d^nx\rangle-\langle\phi^{a_1}_{x_1}\cdots\phi^{a_k}_{x_k}\rangle\langle\int\frac{\partial\widehat u^a_x}{\partial\phi^a_x}d^nx\rangle=0.$$

A glance at the expressions (82) – (83) shows that the Ward identities contain the anomaly, in general, because the measure terms of symbolic functional integrals need not be BRST invariant. If $\mathrm{Sp}(\widehat u)$ is either a finite or infinite number, the Ward identities

$$\langle\widehat u(\phi^{a_1}_{p_1}\cdots\phi^{a_k}_{p_k})\rangle=\sum_{i=1}^{k}(-1)^{[a_1]+\cdots+[a_{i-1}]}\langle\phi^{a_1}_{p_1}\cdots\phi^{a_{i-1}}_{p_{i-1}}\widehat u^{a_i}_{p_i}\phi^{a_{i+1}}_{p_{i+1}}\cdots\phi^{a_k}_{p_k}\rangle=0, \tag{84}$$
$$\langle\widehat u(\phi^{a_1}_{x_1}\cdots\phi^{a_k}_{x_k})\rangle=\sum_{i=1}^{k}(-1)^{[a_1]+\cdots+[a_{i-1}]}\langle\phi^{a_1}_{x_1}\cdots\phi^{a_{i-1}}_{x_{i-1}}\widehat u^{a_i}_{x_i}\phi^{a_{i+1}}_{x_{i+1}}\cdots\phi^{a_k}_{x_k}\rangle=0 \tag{85}$$

are free of this anomaly.

Clearly, all the expressions (82) – (85) are singular, unless regularization and renormalization procedures are involved. In the present work, we concern neither these procedures nor the Wick rotation of the Ward identities (82) – (83).

## XI. EXAMPLE. SUPERSYMMETRIC YANG–MILLS THEORY

Let $\mathcal G=\mathcal G_0\oplus\mathcal G_1$ be a finite-dimensional real Lie superalgebra with a basis $\{e_r\}$, $r=1,\ldots,m$, and real structure constants $c^r_{ij}$. For the sake of simplicity, the Grassmann parity of $e_r$ is denoted $[r]$. Recall the standard relations

$$c^r_{ij}=-(-1)^{[i][j]}c^r_{ji},\qquad [r]=[i]+[j],$$
$$(-1)^{[i][b]}c^r_{ij}c^j_{ab}+(-1)^{[a][i]}c^r_{aj}c^j_{bi}+(-1)^{[b][a]}c^r_{bj}c^j_{ia}=0.$$

Let us also introduce the modified structure constants

$$\overline c^r_{ij}=(-1)^{[i]}c^r_{ij},\qquad \overline c^r_{ij}=(-1)^{([i]+1)([j]+1)}\overline c^r_{ji}. \tag{86}$$

Given the universal enveloping algebra $\overline{\mathcal G}$ of $\mathcal G$, we assume that there is an invariant even quadratic element $h^{ij}e_ie_j$ of $\overline{\mathcal G}$ such that the matrix $h^{ij}$ is non-degenerate. All Lagrangians are further considered up to $d_H$-exact terms.

The Yang–Mills theory of gauge potentials on $X=\mathbb{R}^n$ associated to the Lie superalgebra $\mathcal{G}$ is described by the BGDA $\mathcal{P}^*[Q,Y]$ where

$$Q=(X\times\mathcal{G}_1)\underset{X}{\otimes}T^*X, \qquad Y=(X\times\mathcal{G}_0)\underset{X}{\otimes}T^*X.$$

Its basis is $\{a^r_\lambda\}$, $[a^r_\lambda]=[r]$. There is the canonical decomposition of the first jets of its elements

$$a^r_{\lambda\mu}=\frac{1}{2}(\mathcal{F}^r_{\lambda\mu}+\mathcal{S}^r_{\lambda\mu})=\frac{1}{2}(a^r_{\lambda\mu}-a^r_{\mu\lambda}+c^r_{ij}a^i_\lambda a^j_\mu)+\frac{1}{2}(a^r_{\lambda\mu}+a^r_{\mu\lambda}-c^r_{ij}a^i_\lambda a^j_\mu).$$

Then the Euclidean Yang–Mills Lagrangian takes the form

$$L_{YM}=\frac{1}{4}h_{ij}\eta^{\lambda\mu}\eta^{\beta\nu}\mathcal{F}^i_{\lambda\beta}\mathcal{F}^j_{\mu\nu}d^nx,$$

where $\eta$ is the Euclidean metric on $\mathbb{R}^n$. Its variational derivatives $\mathcal{E}^\lambda_r$ obey the irreducible Noether identities

$$-c^r_{ji}a^i_\lambda\mathcal{E}^\lambda_r-d_\lambda\mathcal{E}^\lambda_j=0.$$

Therefore, we enlarge the BGDA $\mathcal{P}^*[Q,Y]$ to the BGDA

$$P^*\{0\}=\mathcal{P}^*[\overline{E}^*V\overline{Y}^*;Q;Y;\overline{Q}^*E\overline{V}^*], \qquad V=X\times\mathcal{G}_1, \qquad E=X\times\mathcal{G}_0, \tag{87}$$

whose basis

$$\{a^r_\lambda,c^r,\overline{a}^\lambda_r,\overline{c}_r\}, \qquad [c^r]=([r]+1)\mathrm{mod}\,2, \qquad [\overline{a}^\lambda_r]=[\overline{c}_r]=[r],$$

consists of gauge potentials $a^r_\lambda$, ghosts $c^r$ of ghost number 1, and antifields $\overline{a}^\lambda_r$, $\overline{c}_r$ of antifield numbers 1 and 2, respectively. Then, the Noether operators $\Delta_r$ (15), the total gauge operator $u_e$ (35), and the Lagrangian $L_e$ (28) read

$$\begin{aligned}
&\Delta_j=-c^r_{ji}a^i_\lambda\overline{a}^\lambda_r-\overline{a}^\lambda_{\lambda j},\\
&u_e=u^r_\lambda\partial^\lambda_r=(-c^r_{ji}c^ja^i_\lambda+c^r_\lambda)\partial^\lambda_r,\\
&L_e=L_{YM}-c^j(c^r_{ji}a^i_\lambda\overline{a}^\lambda_r+\overline{a}^\lambda_{\lambda j})d^nx=L_{YM}+(-c^r_{ji}c^ja^i_\lambda+c^r_\lambda)\overline{a}^\lambda_rd^nx+d_H\sigma.
\end{aligned}$$

The total gauge operator $u_e$ admits the nilpotent extension

$$u_E=u_e+\xi=(-c^r_{ji}c^ja^i_\lambda+c^r_\lambda)\partial^\lambda_r-\frac{1}{2}\overline{c}^r_{ij}c^ic^j\partial_r,$$

where $\overline{c}^r_{ij}$ are the modified structure constants (86). Then, the extended Lagrangian

$$L_E=L_{YM}+(-c^r_{ij}c^ja^i_\lambda+c^r_\lambda)\overline{a}^\lambda_rd^nx-\frac{1}{2}\overline{c}^r_{ij}c^ic^j\overline{c}_rd^nx, \tag{88}$$

obeys the master equation.

Passing to the gauge-fixing Lagrangian, we enlarge the BGDA $P^*[0]$ (87) to the BGDA

$$\overline{P}^*\{0\} = \mathcal{P}^*[\overline{E}^*\overline{V}^*V\overline{Y}^*; Q; Y; \overline{Q}^*E\overline{E}^*\overline{V}^*],$$

possessing the basis

$$\{a^r_\lambda, c^r, c^*_r, \overline{a}^\lambda_r, \overline{c}_r\}, \qquad [c^*_r] = [c^r].$$

Let us choose the gauge-fixing fermion

$$\Psi = \frac{1}{2}\eta^{\lambda\mu}\mathcal{S}^r_{\lambda\mu}c^*_r = \eta^{\lambda\mu}a^r_{\lambda\mu}c^*_r,$$

and replace the antifields in the extended Lagrangian $L_E$ (88) with the terms

$$\overline{a}^\lambda_r = \frac{\delta\Psi}{\delta a^r_\lambda} = -\eta^{\lambda\mu}c^*_{\mu r}, \qquad \overline{c}_r = 0.$$

We come to the Lagrangian $L_\Psi$ (56) which reads

$$L_\Psi = L_{YM} - \eta^{\lambda\mu}(-c^r_{ij}c^j a^i_\lambda + c^r_\lambda)c^*_{\mu r}d^nx = L_{YM} + (-1)^{[r]+1}\eta^{\lambda\mu}c^*_r d_\mu(-c^r_{ij}c^j a^i_\lambda + c^r_\lambda)d^nx + d_H\sigma.$$

It is brought into the form

$$L_\Psi = L_{YM} + c^*_r M^r_j c^j d^nx,$$

where

$$M^r_i = (-1)^{[r]+1}\eta^{\lambda\mu}(c^r_{ij}(a^i_{\mu\lambda} + a^i_\lambda d_\mu) + \delta^r_j d_{\mu\lambda})$$

is a second order differential operator acting on the ghosts $c^j$. Finally, we write the gauge-fixing Lagrangian

$$\begin{aligned}L_{GF} = L_{YM} + c^*_r M^r_j c^j d^nx + \frac{1}{8}h_{ij}\eta^{\lambda\mu}\eta^{\beta\nu}\mathcal{S}^i_{\lambda\mu}\mathcal{S}^j_{\beta\nu}d^nx = \\ L_{YM} + [(-1)^{[r]+1}\eta^{\lambda\mu}c^*_r d_\mu(-c^r_{ij}c^j a^i_\lambda + c^r_\lambda) + \frac{1}{2}h_{ij}\eta^{\lambda\mu}\eta^{\beta\nu}a^i_{\lambda\mu}a^j_{\beta\nu}]d^nx,\end{aligned}$$

which possesses the BRST symmetry

$$\widehat{u} = (-c^r_{ij}c^j a^i_\lambda + c^r_\lambda)\frac{\partial}{\partial a^r_\lambda} - \frac{1}{2}\overline{c}^r_{ij}c^i c^j\frac{\partial}{\partial c^r} + (-1)^{[j]}h_{ij}\eta^{\lambda\mu}a^i_{\lambda\mu}\frac{\partial}{\partial c^*_j}.$$

Quantizing this Lagrangian system in the framework of Euclidean perturbed QFT , we come to the graded commutative tensor algebra $B_\Phi$ generated by the elements $\{a^r_{x\lambda}, c^r_x, c^*_{xr}\}$. Its states $\langle\phi\rangle$, $\phi\in B_\Phi$, are given by functional integrals

$$\langle\phi\rangle = \frac{1}{\mathcal{N}}\int \phi\exp\{-\int \mathcal{L}_{GF}(a^r_{x\Lambda\lambda}, c^r_{x\Lambda}, c^*_{xr})d^nx\}\prod_x [da^r_{x\lambda}][dc^r_x][dc^*_{xr}],$$

$$\mathcal{N} = \int\exp\{-\int \mathcal{L}_{GF}(a^r_{x\Lambda\lambda}, c^r_{x\Lambda}, c^*_{xr})d^nx\}\prod_x [da^r_{x\lambda}][dc^r_x][dc^*_{xr}],$$

$$\mathcal{L}_{GF} = \mathcal{L}_{YM} + (-1)^{[r]+1}\eta^{\lambda\mu}c^*_{xr}d_\mu(-c^r_{xij}c^j_x a^i_{x\lambda} + c^r_{x\lambda}) + \frac{1}{2}h_{ij}\eta^{\lambda\mu}\eta^{\beta\nu}a^i_{x\lambda\mu}a^j_{x\beta\nu}.$$

Accordingly, the quantum BRST transformation (81) reads

$$\widehat{u} = \int[(-c^r_{ij}c^j_x a^i_{x\lambda} + c^r_{x\lambda})\frac{\partial}{\partial a^r_{x\lambda}} - \frac{1}{2}\overline{c}^r_{ij}c^i_x c^j_x\frac{\partial}{\partial c^r_x} + (-1)^{[j]}h_{ij}\eta^{\lambda\mu}a^i_{x\lambda\mu}\frac{\partial}{\partial c^*_{xj}}]d^nx.$$

It is readily observed that $\mathrm{Sp}(\widehat{u}) = 0$. Therefore, we obtain the Ward identities $\langle\widehat{u}(\phi)\rangle = 0$ (85) without anomaly.


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